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Self-Avoiding Walks and Trees in Spread-Out Lattices

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Let $\mathscr{G}_R$ be the graph obtained by joining all sites of Z^d which are separated by a distance of at most R. Let $\mu(\mathscr{G}_R)$ denote the connective constant for counting the self-avoiding walks in this graph. Let $\lambda(\mathscr{G}_R)$ denote the coprresponding constant for counting the trees embedded in $\mathscr{G}_R$. Then as $R \to \infty$, $\mu(\mathscr{G}_R)$ is asymptotic to the coordination number k_R of $\mathscr{G}_R$, while $\lambda(\mathscr{G}_R)$ is asymptotic to ek_R. However, if d is 1 or 2, then $\mu(\mathscr{G}_R) - k_R$ diverges to $-\infty$.

KEY WORDS: Self-avoiding random walk; connective constant; mean-field behavior; trees; polymers.

1. INTRODUCTION

The following quantities arise in connection with the study of polymers with excluded volume effects. For a periodic graph in d dimensions, i.e., a translation-invariant graph $\mathscr{G}$ with vertex set Z^d, let $c_n(\mathscr{G})$ denote the number of n-step *self-avoiding walks* in the graph $\mathscr{G}$ starting from the origin. See, e.g., Madras and Slade.[16] Let $t_n(\mathscr{G})$ denote the number of distinct *trees* with n edges and with the origin as a vertex, which can be obtained as subgraphs of $\mathscr{G}$. A lattice tree can be used as a model for a *branched* polymer; see refs. 19 and 12 and references therein.

The sequence $c_n(\mathscr{G})$ is submultiplicative (i.e., $c_{n+m} \leqslant c_n c_m$ for all n, m). Also, $(n^{-1} t_n(\mathscr{G}))$ is a *super*multiplicative sequence,[12] and by well-known arguments[5,16] there exist limits

$$\mu(\mathscr{G}) := \lim_{n \to \infty} (c_n(\mathscr{G}))^{1/n} = \inf_n (c_n(\mathscr{G}))^{1/n} \tag{1}$$

$$\lambda(\mathscr{G}) := \lim_{n \to \infty} (t_n(\mathscr{G}))^{1/n} \tag{2}$$

Dedicated to Oliver Penrose on this occasion of his 65th birthday.

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(the symbol := denotes definition). The constants $\mu(\mathscr{G})$ and $\lambda(G)$ are measures of the richness of the connective structure of the latice $\mathscr{G}$, as is the critical point for bond percolation on $\mathscr{G}$, which we denote $p_c(\mathscr{G})$.

When $\mathscr{G}$ is the two-dimensional hexagonal lattice, Nienhuis[17] has argued (nonrigorously) that $\mu(G) = (2+\sqrt{2})^{1/2}$. Aside from this case, the exact values of $\mu(\mathscr{G})$ and $\lambda(\mathscr{G})$ are unknown for nontrivial graphs $\mathscr{G}$. This fact has motivated the study of these quantities for the nearest-neighbour graph Z^d in the mean-field limit $d \to \infty$.[2,6,9–11] For example, it is known[11,9] that as $d \to \infty$,

$$\mu(Z^d) = 2d - 1 - (2d)^{-1} - 3(2d)^{-2} - 16(2d)^{-3} - 102(2d)^{-4} + O(d^{-5}) \quad (3)$$

In this paper we consider a different mean-field limit, one which is easier to interpret physically. For $R > 0$, let $\mathscr{G}_R$ denote the graph on Z^d whose edges are those (x, y) with $0 < \|x - y\| \leqslant R$. Here $\|\cdot\|$ denotes an arbitrary fixed norm on $\mathscr{R}^d$. We shall consider the quantities $\mu(\mathscr{G}_R)$ and $\lambda(\mathscr{G}_R)$ in the limit $R \to \infty$ with d fixed. We shall call this the *spread-out limit*. It is analogous to the "van der Waals limit" (or "Kac limit") considered by Lebowitz and Penrose.[15]

Let $\mathscr{T}_k$ denote the complete (rooted, labeled) k-ary tree (Bethe lattice).[3,14] Let $b_b(k)$ denote the number of subtrees θ of $\mathscr{T}_k$ with n edges (and hence with $n+1$ nodes) such that one of the nodes of θ is the root of $\mathscr{T}_k$. Then[14]

$$b_n(k) = \binom{k(n+1)}{n+1} \frac{1}{(k-1)(n+1)+1} \quad (4)$$

By Stirling's formula,

$$\lim_{n \to \infty} (b_n(k))^{1/n} = \frac{k^k}{(k-1)^{k-1}} := \tau(k) \quad (5)$$

Observe that $\tau(k) \sim ek$ as $k \to \infty$. Here and below, the symbol $\sim$ means the ratio of the two sides approaches 1.

For any periodic graph $\mathscr{G}$ let $k(\mathscr{G})$ denote the coordination number (i.e., the degree of a given vertex), so, for example, if $d = 1$ with the usual norm, then $k(\mathscr{G}_R) = 2[R]$. One has mean-field upper bounds for $\mu(\mathscr{G})$ and $\lambda(\mathscr{G})$ by the corresponding quantities for the complete $k(\mathscr{G})$-ary tree:

$$\mu(\mathscr{G}) \leqslant k(\mathscr{G}); \qquad \lambda(\mathscr{G}) \leqslant \tau(k(\mathscr{G})) \quad (6)$$

We shall prove here that in the spread-out limit, $\mu(\mathscr{G}_R)$ and $\lambda(\mathscr{G}_R)$ are asymptotic to these upper bounds. We use the abbreviation $k_R := k(\mathscr{G}_R)$.

Also, c denotes a positive finite constant which may vary from line to line (not to be confused with c_n).

Theorem 1. Let $d \geqslant 1$. Then for some constant $c > 0$ and some $R_0 > 0$,

$$k_R \geqslant \mu(\mathscr{G}_R) \geqslant k_R - c k_R^{5/7} \log(k_R), \qquad R \geqslant R_0 \tag{7}$$

In particular, $\mu(\mathscr{G}_R) \sim k_R$ as $R \to \infty$.

Theorem 2. Let $d \geqslant 1$. Then for some constant $c > 0$ and some $R_0 > 0$,

$$\tau(k_R) \geqslant \lambda(\mathscr{G}_R) \geqslant \tau(k_R) - c k_R^{5/7} \log(k_R), \qquad R \geqslant R_0 \tag{8}$$

In particular, $\lambda(\mathscr{G}_R) \sim \tau(k_R)$ as $R \to \infty$.

Some cases of Theorem 1 were already known. For $d \geqslant 2$, the spread-out limit for the bond percolation threshold $p_c(\mathscr{G})$ was considered in ref. 18, where the asymptotic result $p_c(\mathscr{G}_R) \sim 1/k_R$ was established. The result $\mu(\mathscr{G}_R) \sim k_R$ (for $d \geqslant 2$) is immediate from this, together with (6) and the inequality[4]

$$\mu(\mathscr{G}) \geqslant 1/p_c(\mathscr{G}) \tag{9}$$

Also, for $d > 4$, Madras and Slade (ref. 16, Corollary 6.2.7) have a better lower bound than (7), namely that for any $s > 0$ there exists $c > 0$ such that $\mu(\mathscr{G}_R) \geqslant k_R - c k_R^{s + 2/d}$ for all large enough R. However, the proof here is simpler than either of these, and works for all $d \geqslant 1$.

By analogy with Eq. (3) one might aim in the spread-out limit for an expansion of $\mu(\mathscr{G}_R) - k_R$ in powers of k_R. For $d \leqslant 2$, the next result shows that any such expansion must have as leading term a positive power of k_R.

Theorem 3. There are positive constants c and R_0 such that for $R \geqslant R_0$,

$$\mu(\mathscr{G}_R) \leqslant k_R - c k_R^{1/5} \qquad \text{if} \quad d = 1 \tag{10}$$

$$\mu(\mathscr{G}_R) \leqslant k_R - c \log(k_R) \qquad \text{if} \quad d = 2 \tag{11}$$

For the case $d \geqslant 3$, we conjecture that $k_R - \mu(\mathscr{G}_R)$ converges in the spread-out limit to the expected total number of vists to the unit ball (of $\|\cdot\|$) by a random walk in $\mathscr{R}^d$ with steps uniformly distributed over the unit ball. When $d > 4$, the lace expansion developed by Hara and Slade[6–8] should be highly relevant to this problem. See in particular ref. 16, Chapter 6.

Note that from (11), together with (9), we have for $d=2$ that for some constant c and all large enough R,

$$p_c(\mathcal{G}_R) \geqslant (1/k_R) + c(\log k_R)/k_R^2 \tag{12}$$

The next section contains the proofs. Section 3 will describe related models for which similar results hold. These include lattice annimals, and models with excluded volume effects along the edges of the walk or tree, not just at the vertices. These last models exhibit dimension dependence not apparent in Theorems 1 and 2.

2. PROOFS

In these proofs we shall for convenience take $\|\cdot\|$ to be the L^∞-norm $\|(x^1,..., x^d)\| = \max_i |x^i|$. The modifications to other norms are straightforward for Theorems 1 and 2, and should also be true for Theorem 3 [see the remark following Eq. (40) below]. Let $\mathcal{G}'_R$ denote the graph with vertex set $Z^d/R := \{x/R\colon x \in Z^d\}$ and with edge set $\{(z_1, z_2)\colon 0 < \|z_1 - z_2\| \leqslant 1\}$. This is clearly isomorphic to $\mathcal{G}_R$.

Let $V(i)$, $i = 1, 2, 3,...$, denote independent identically distributed (i.i.d.), $\mathcal{R}^d$-valued random variables with $V(i)$ uniformly distributed on the unit ball of $\|\cdot\|$. Similarly, for $R \geqslant 1$ let $V_R(i)$, $i = 1, 2, 3,...$ denote i.i.d. discrete variables which are uniformly distributed on $\{z \in Z^d/R\colon 0 < \|z\| \leqslant 1\}$, the sites joined to 0 in $\mathcal{G}'_R$. Define the random walk paths

$$S(m) = \sum_{i=1}^{m} V(i), \qquad S_R(m) = \sum_{i=1}^{m} V_R(i), \qquad m = 1, 2, 3,... \tag{13}$$

Given an open set in $\mathcal{R}^d$, for R large the probability that $S_R(n)$ lies in that set decays rather slowly as n becomes large, by the Local Limit Theorem. This idea is the basis of the following lemma.

Lemma 1. For $j = 0, 1, 2,...$ let A_j denote the slab

$$A_j := \{(x^1,..., x^d) \in \mathcal{R}^d\colon j - (1/2) < x^1 \leqslant j + (1/2)\} \tag{14}$$

Then there exist $R_1 \geqslant 3$ and $\gamma \in (0, 1]$ such that for all $R \geqslant R_1$,

$$P[x_0 + S_R(m) \in A_{j+1}] \geqslant 2\gamma m^{-1/2} \qquad \text{for all} \quad x_0 \in A_j, \quad m \geqslant 1 \tag{15}$$

Remark. We take $R_1 \geqslant 3$ so as to use the comment after (18) below.

Proof. It suffices to consider the case $j = 0$. Let K be an integer with $K \geqslant 98\pi^{1/2}$. Write V_R in coordinates as $(V_R^1,..., V_R^d)$. Similarly, for $1 \leqslant i \leqslant d$ let V^i, S_R^i, and x_0^i denote the ith coordinate of V, S_R, and x_0 respectively.

For all R, $E[V_R^1]=0$. Set $\sigma_R^2 := E[|V_R^1|^2]$ and $\rho_R := E[|V_R^1|^3]$. Since $E[|V^1|^2] = 1/3$, there exists $R_1 \geqslant 3$ such that $\frac{1}{4} \leqslant \sigma_R^2 \leqslant \frac{1}{2}$ for all $R \geqslant R_1$. Also, $\rho_R \leqslant 1$ for all R. Using these estimates and the Berry–Esseen theorem[1] and setting Φ to be the standard normal distribution function, we can find $m_0 \geqslant 1$ such that for all $x \in A_0$, $m \geqslant m_0$, and $R \geqslant R_1$,

$$P[x_0^1 + S_R^1(m) \in (0, K]] \geqslant \Phi\left(\frac{K - x_0^1}{\sigma_R m^{1/2}}\right) - \Phi\left(\frac{-x_0^1}{\sigma_R m^{1/2}}\right) - \frac{6\rho_R}{\sigma_R^3 m^{1/2}}$$
$$\geqslant \frac{K}{2(\pi m)^{1/2}} - \frac{48}{m^{1/2}} \geqslant m^{-1/2}$$

By a tightness argument, there is a constant $\gamma_0 > 0$ such that for all $y \in [0, K]$ and all $R \geqslant 3$,

$$P[y + S_R^1(2K) \in (1/2, 3/2]] \geqslant \gamma_0 \tag{16}$$

Therefore, there exists $\gamma_1 > 0$ such that for all $R \geqslant R_1$,

$$\begin{aligned} P[x_0 + S_R(m) \in A_1] &\geqslant P[x_0^1 + S_R^1(m - 2K) \in [0, K]] \times \gamma_0 \\ &\geqslant \gamma_0 (m - 2K)^{-1/2} \\ &\geqslant \gamma_1 m^{-1/2} \qquad \text{for} \quad x_0 \in A_0, \quad m \geqslant 3K \end{aligned} \tag{17}$$

Again using tightness, one sees $P[x_0 + S_R(m) \in A_1]$ is uniformly bounded away from zero on $\{x_0 \in A_0,\ R \geqslant 3,\ 1 \leqslant m < 3K\}$, and so with a change of constant, the lower bound (17) holds over *all* $m \geqslant 1$, as desired. ∎

Before giving details, we sketch the proof of Theorem 1. Fix $\varepsilon > 0$. By Lemma 1 we can find m_1 so large that for any large R, if F_j denotes the event that $S_R(im_1) \in A_i$ for $i = 1, 2,..., j$, then $P[F_j] \geqslant (1-\varepsilon)^{jm_1}$, uniformly in j. Now fix m_1 also. The conditional probability, given F_j, that the random walk $(S_R(1),..., S_R(jm_1))$ is self-avoiding can be made to exceed $(1-\varepsilon)^{jm_1}$ (uniformly in j) by making R big. This is because the occurrence of F_j ensures that only pieces of the path $S_R(n)$ and $S_R(m)$ with $|n-m| \leqslant 4m_1^2$ can possibly intersect {consider the distance between $S_R(m_1[m/m_1])$ and $S_R(m_1[n/m_1])$}, and when R is very big, any two individual pieces are highly unlikely to intersect. The two lower bounds together imply that for large R, the path $S_R(n)$, $1 \leqslant n \leqslant jm_1$, is self-avoiding with probability exceeding $(1-2\varepsilon)^{jm_1}$ (uniformly in j). This translates into a lower bound of $k_R(1-2\varepsilon)$ on $\mu(\mathcal{G}_R)$. The proof of Theorem 2 is more involved, but uses a similar idea.

Proof of Theorem 1. It suffices to prove the second inequality in (7). Define the function

$$f_1(\varepsilon) := (2/\varepsilon)\log(1/\varepsilon), \qquad \varepsilon > 0 \tag{18}$$

Since we took $\gamma \leqslant 1$ in Lemma 1, the function $(\gamma x^{-1/2})^{1/x}$ is increasing in x on $x \geqslant 3$. There exists $\varepsilon_0 > 0$, with $f_1(\varepsilon_0) \geqslant 3$, such that if $0 < \varepsilon \leqslant \varepsilon_0$, then

$$(\gamma x^{-1/2})^{1/x} > 1 - \varepsilon \qquad \text{if} \quad x \geqslant [f_1(\varepsilon)] \tag{19}$$

Now fix $\varepsilon \in (0, \varepsilon_0)$ and set $m_1 := [f_1(\varepsilon)]$, the integer part of $f_1(\varepsilon)$.

For $j = 0,, 1, 2, \ldots$, let A_j be as in Lemma 1, and define the events

$$\begin{aligned} L_j &= L_j(R) := \{S_R(jm_1) \in A_j\} \\ M_j &= M_j(R) := \{S_R(m) \neq S_R(n) \text{ for } 0 \leqslant m < n \leqslant jm_1\} \\ H_j &= H_j(R) := L_j(R) \cap M_j(R) \end{aligned} \tag{20}$$

We estimate the probabilities of these events. By Lemma 1,

$$P\left[L_{j+1}(R) \,\middle|\, \bigcap_{i=1}^{j} H_i(R)\right] \geqslant 2\gamma m_1^{-1/2}, \qquad R \geqslant R_1 \tag{21}$$

Let $R \geqslant R_1$, $j \geqslant 1$. Suppose that $L_1, \ldots, L_j$ all occur, and that for some n with $0 \leqslant n < jm_1$ we have $\|S_R(n) - S_R(jm_1)\| \leqslant m_1$. Write $n = rm_1 + s$ with $r, s \in Z$, $0 \leqslant r < j$, and $0 \leqslant s < m_1$. By the triangle inequality, $\|S_R(rm_1) - S_R(jm_1)\| \leqslant 2m_1$, so by the assumption that L_r and L_j occur, $(j - r - 1) \leqslant 2m_1$, i.e., the total thickness of the slabs A_i lying strictly between $S_R(rm_1)$ and $S_R(jm_1)$ is at most $2m_1$. Therefore, $j - r \leqslant 2m_1 + 1 \leqslant 3m_1$, and so $jm_1 - n \leqslant 3m_1^2$. Therefore,

$$P\left[\operatorname{card}\{n < jm_1 : \|S_R(n) - S_R(jm_1)\| \leqslant m_1\} \leqslant 3m_1^2 \,\middle|\, \bigcap_{i=1}^{j} L_i\right] = 1 \tag{22}$$

For $jm_1 < n \leqslant (j+1)m_1$ and $z \in Z^d/R$, the probability that $S_R(n) = z$, conditional on the history of the walk up to time $n - 1$, is bounded above by $1/k_R$ for $\|z - S_R(jm_1)\| \leqslant m_1$ and by 0 for other z. So by (22), the complement $M_{j+1}^c(R)$ satisfies

$$P\left[M_{j+1}^c \,\middle|\, \bigcap_{i=1}^{j} H_i\right] \leqslant \frac{m_1(3m_1^2) + m_1(m_1+1)/2}{k_R} \leqslant \frac{4m_1^3}{k_R} \tag{23}$$

Set

$$R_2(\varepsilon) := (4/\gamma)^{1/d}(f_1(\varepsilon))^{7/(2d)} = (4/\gamma)^{1/d}((2/\varepsilon)\log(1/\varepsilon))^{7/(2d)} \tag{24}$$

For our choice of norm, $k_R \geqslant (2[R])^d \geqslant R^d$ for $R > 1$. So

$$P\left[M_{j+1}^c(R) \,\middle|\, \bigcap_{i=1}^{j} H_i(R)\right] \leqslant \gamma m_1^{-1/2} \qquad \text{if} \quad R \geqslant R_2(\varepsilon) \tag{25}$$

By (21) and (25), for $R \geqslant R_2(\varepsilon)$ we have

$$P\left[H_{j+1}(R) \,\middle|\, \bigcap_{i=1}^{j} H_i(R)\right] \geqslant 2\gamma m_1^{-1/2} - \gamma m_1^{-1/2} \tag{26}$$

Since the event $\bigcap_{i=1}^{j} H_i(R)$ implies that the random walk S_R is self-avoiding up to time jm_1, we have by induction that for $R \geqslant R_2(\varepsilon)$,

$$\frac{c_{jm_1}(\mathscr{G}_R)}{k_R^{jm_1}} \geqslant P\left[\bigcap_{i=1}^{j} H_i(R)\right] \geqslant (\gamma m_1^{-1/2})^j, \qquad j \geqslant 1 \tag{27}$$

By (19), this implies that

$$(c_{jm_1}(\mathscr{G}_R))^{1/(jm_1)} \geqslant k_R(\gamma m_1^{-1/2})^{1/m_1} \geqslant k_R(1-\varepsilon), \qquad j \geqslant 1 \tag{28}$$

Taking $j \to \infty$, we obtain

$$\mu(\mathscr{G}_R) \geqslant k_R(1-\varepsilon), \qquad R \geqslant R_2(\varepsilon) \tag{29}$$

Now $R_2(\varepsilon)$ is a one-to-one function of ε. Set $\varepsilon(R)$ to be its inverse. Then for suitable constants c and c', for all $R \geqslant R_2(\varepsilon_0)$,

$$\varepsilon(R) \leqslant cR^{-2d/7} \log R \leqslant c'k_R^{-2/7} \log k_R \tag{30}$$

Using (29), we have for $R \geqslant R_2(\varepsilon_0)$ that

$$\mu(\mathscr{G}_R) \geqslant k_R(1-\varepsilon(R)) \geqslant k_R - c'k_R^{5/7} \log(k_R) \quad \blacksquare$$

Proof of Theorem 2. First we write down a stronger version of Eq. (5). By the definition (4) of $b_n(k)$ and Stirling's formula, one can check that there are constants $c < c'$ such that for all $n \geqslant 1$ and $k \geqslant 2$,

$$(cn^{-3/2})^{1/n} \leqslant (b_n(k))^{1/n}/\tau(k) \leqslant (c'n^{-3/2})^{1/n} \tag{31}$$

By the first of these inequalities, there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that if we fix $\varepsilon \in (0, \varepsilon_1]$ and set $m_1 = [(2/\varepsilon)\log(1/\varepsilon)]$ as before, then

$$(b_{m_1}(k))^{1/m_1} \geqslant \tau(k)(1-\varepsilon) \qquad \text{for all} \quad k \geqslant 2 \tag{32}$$

We can identify nodes of the complete k-ary tree $\mathscr{T}_k$ with words $a_1 \cdots a_n$ from the alphabet $\{1,..., k\}$.[3] The letter a_i tells us which branch to take at depth i when traveling down from the root of $\mathscr{T}_k$ to the node $a_1 \cdots a_n$. The root of $\mathscr{T}_k$ is represented by the empty word $\varnothing$. Given any word (node) w of $\mathscr{T}_k$, let $|w|$ denote the depth of w, i.e., the number of letters in w. Also, let $\Theta_n^k(w)$ denote the set of all subtrees θ of $\mathscr{T}_k$ such that (i) θ has n edges, (ii) w is a node of θ, and (iii) $|w| \leqslant |w'|$ for all nodes w' of θ (i.e., θ is rooted at w). For any node w of $\mathscr{T}_k$ the cardinality of $\Theta_n^k(w)$ is $b_n(k)$ as given by (4).

Choose an ordering on the sites of $\mathscr{G}'_R$ joined to 0, i.e., write $\{z \in Z^d/R: 0 < \|z\| \leqslant 1\}$ as $\{z_1,..., z_{k_R}\}$. Define a mapping F of $\mathscr{T}_{k_R}$ onto Z^d/R by $F(\varnothing) = 0$ and

$$F(a_1 \cdots a_n) = z_{a_1} + \cdots + z_{a_n} \tag{33}$$

This induces a mapping (also denoted F) from subtrees of $\mathscr{T}_{k_R}$ to connected subgraphs (not always trees) of $\mathscr{G}'_R$. The restriction of F to subtrees of $\mathscr{T}_{k_R}$ which have the empty word as their root and which are mapped by F to *trees* in $\mathscr{G}'_R$ is one-to-one.

Define a random algorithm as follows. Let $w(0)$ denote the empty word $\varnothing$, i.e., the root of $\mathscr{T}_{k_R}$. The first step of the algorithm is to let U_1 be a subtree of $\mathscr{T}_{k_R}$ chosen randomly from $\Theta_{m_1}^{k_R}(w(0))$. Let $q_1 := \max_{w \in U_1}(|w|)$ be the depth of U_1. Then let $w(1)$ be chosen uniformly at random from the set of nodes w of U_1 with $|w| = q_1$ (the set of nodes of maximal depth in U_1), and let $X_1 := F(w(1))$. We shall deem the first step to be a "success" if (i) the image $F(U_1)$ of U_1 under F is a tree in $\mathscr{G}'_R$, and (ii) $X_1 \in A_1$.

Subsequent steps in the algorithm are defined inductively as follows. Assume steps 1 to j havbe been successful; there will then be defined a word $w(j)$ of $\mathscr{T}_{k_R}$ with $X_j := F(w(j)) \in A_j$. We then define the $(j+1)$th step as follows. Let U_{j+1} be a subtree of $\mathscr{T}_{k_R}$ chosen at random from $\Theta_{m_1}^{k_R}(w(j))$ [a set of cardinality $b_{m_1}(k_R)$]. Let $q_{j+1} := \max_{w \in U_{j+1}}(|w| - |w(j)|)$ be the depth of U_{j+1} (so, clearly, $q_{j+1} \leqslant m_1$). Let $w(j+1)$ be chosen uniformly at random from the set of nodes w of U_{j+1} with $|w| - |w(j)| = q_{j+1}$. Let $X_{j+1} := F(w_{j+1})$. Deem the $(j+1)$th step to be a "success" if (i) the image $F(U_{j+1})$ of U_{j+1} in $\mathscr{G}'_R$ is a tree, (ii) $F(U_{j+1})$ has no common vertices with any of $F(U_1),..., F(U_j)$, except for $w(j)$, and (iii) $X_{j+1} \in A_{j+1}$.

Let L'_j denote the event $\{X_j \in A_j\}$. Let M'_j denote the event that the image under F of the tree $\bigcup_{i=1}^{j} U_i$ is a tree, i.e., no two sites of $\bigcup_{i=1}^{j} U_i$ are mapped by F to the same site of $\mathscr{G}'_R$. Finally, define the event $H'_j := L'_j \cap M'_j$ (these events also depend on R). Given success up to step j, the event H'_{j+1} is equivalent to success at step $j+1$, and we now estimate the probability of success.

Suppose steps 1 to j are successful. Conditional on the identities of $U_1,..., U_j$ and on the value of q_{j+1}, the path from $F(X_j)$ to $F(X_{j+1})$ traced out by the image under F of the path along $\mathscr{T}_{k_R}$ from $w(j)$ to $w(j+1)$ is equally likely to be any of the possible q_{j+1}-step random walk paths in $\mathscr{G}'_R$ starting from X_j (which lies in the slab A_j). By Lemma 1, since $q_{j+1} \leqslant m_1$,

$$P\left[L'_{j+1} \,\Big|\, \bigcap_{i=1}^{j} H'_i\right] \geqslant 2\gamma m_1^{-1/2}, \qquad R \geqslant R_1 \tag{34}$$

We now turn to M'_{j+1}. By a similar argument to the one which led to (23) in the proof of Theorem 1 (we omit details this time),

$$\begin{aligned} P\left[M'^{c}_{j+1} \,\Big|\, \bigcap_{i=1}^{j} H'_i\right] &\leqslant \text{const} \times m_1^3/k_R \\ &\leqslant \gamma m_1^{-1/2} \qquad \text{if} \quad R \geqslant R_3(\varepsilon) \end{aligned} \tag{35}$$

where for some suitably chosen constant c,

$$R_3 = R_3(\varepsilon) := c((1/\varepsilon)\log(1/\varepsilon))^{7/2d} \tag{36}$$

Combining (35) with (34), we have a lower bound of $\gamma m_1^{-1/2}$ for the probability of success at step $j+1$, given success at earlier steps. By induction,

$$P\left[\bigcap_{i=1}^{j} H'_i\right] \geqslant (\gamma m_1^{-1/2})^j \qquad \text{for all} \quad R \geqslant R_3, \quad j \geqslant 1 \tag{37}$$

Now, each successful sequence $U_1,..., U_j$ determines a tree $\hat{T} := \bigcup_{i=1}^{j} F(U_i)$ in G'_R with jm_1 edges. Given $\hat{T}$, one can recover $U_1, U_2,..., U_j$ as follows. Let $\hat{U}$ be the (unique) subtree of $\mathscr{T}_{k_R}$ such that $F(\hat{U}) = \hat{T}$. Take the "trunk" of $\hat{U}$ to be the longest path along $\hat{U}$ starting at the root, then cut the trunk at $j-1$ points so as to divide $\hat{U}$ into j subtrees, each with m_1 edges. These subtrees are $U_1,..., U_j$. Therefore, no two distinct successful sequences can give rise to the same tree $\hat{T}$, and the number of successful sequences $(U_1,..., U_j)$ is at most $t_{jm_1}(G_R)$.

Since at each step, the tree U_j is chosen at random from a set of cardinality $b_{m_1}(k_R)$ [with further randomness due to the choice of $w(j)$], the probability of a particular successful sequence $U_1,..., U_j$ is at most $(b_{m_1}(k_R))^{-j}$. Therefore for all j and $R \geqslant R_3$,

$$(b_{m_1}(k_R))^{-j}\, t_{jm_1}(\mathscr{G}_R) \geqslant P\left[\bigcap_{i=1}^{j} H'_i\right] \geqslant (\gamma m_1^{-1/2})^j \tag{38}$$

By (32) and (19), this implies

$$(t_{jm_1}(\mathscr{G}_R))^{1/(jm_1)} \geqslant (1-2\varepsilon)\,\tau(k_R) \tag{39}$$

The proof is completed by a repetition of the argument following Eq. (28) in the proof of Theorem 1, since $R_3(\varepsilon)$ differs from $R_2(\varepsilon)$ only by a multiplicative constant. ▮

Proof of Theorem 3. For $x \in \mathscr{R}^d$, let $B(x)$ denote the $\|\cdot\|$ unit ball centered at x. Let $(S'_R(m))$ denote the same random walk as $(S_R(m))$ given by (13), except that for S'_R, steps of $\mathbf{0}$ are to be allowed in the walk, i.e., set $S'_R(m) := \sum_{i=1}^m V'_R(i)$, with $V'_R(i)$ independent and uniform over $\{z \in Z^d/R\colon \|z\| \leqslant 1\}$. The point of this is that for $d > 1$ the components of $S'_R(m)$ are independent.

The statement of Lemma 1 still holds with S_R replaced by S'_R and A_{j+1} replaced by A_j in (15). So the following lower bound holds for $d = 1$, and then for all d by independence of components:

$$P[S'_R(m) \in B(0)] \geqslant 2h_d m^{-d/2}, \qquad R \geqslant R_1, \quad m \geqslant 1 \tag{40}$$

where $2h_d := (2\gamma)^d > 0$. This trickery would not work if $\|\cdot\|$ were not the L^∞-norm, but (40) should still be true (with a suitable h_d) for other norms.

For $j \geqslant 0$, let $E_j = E_j(R)$ denote the event that $S'_R(m) \neq S'_R(n)$ for all distinct $m, n \in \{0, 1, 2,..., j\}$. Then

$$\begin{aligned} P[E^c_{j+1} \mid E_j] &= (k_R + 1)^{-1}\, E[\operatorname{card}\{i \leqslant j\colon S'_R(i) \in B(S'_R(j))\} \mid E_j] \\ &= (k_R + 1)^{-1} \left(1 + \sum_{i=1}^{j} \left(\frac{P[\{S'_R(j-i) \in B(S'_R(j))\} \cap E_j]}{P[E_j]}\right)\right) \\ &\geqslant (k_R + 1)^{-1} \sum_{i=1}^{j} (P[S'_R(j-i) \in B(S'_R(j))] - P[E^c_j]) \\ &= (k_R + 1)^{-1} \sum_{i=1}^{j} (P[S'_R(i) \in B(0)] - P[E^c_j]) \end{aligned} \tag{41}$$

Let $n \geqslant 0$, and set

$$R_4 = R_4(n, d) := \left(\frac{n^{2 + d/2}}{h_d}\right)^{1/d} \tag{42}$$

Then for $R \geqslant R_4$ and $j \leqslant n$,

$$P[E^c_j] \leqslant n^2/k_R \leqslant h_d n^{-d/2} \tag{43}$$

Assume n is big enough so that R_4 exceeds the R_1 of Lemma 1. By (41), (40), and (43), there exists a constant c such that for all $R \geqslant R_4$ and $j \leqslant n$,

$$P[E^c_{j+1} \mid E_j] \geqslant (1 + k_R)^{-1} \sum_{i=1}^{j} h_d i^{-d/2} \geqslant c k_R^{-1} g_d(j) \tag{44}$$

where we set $g_d(x)=x^{1/2}$ for $d=1$ and $g_d(x)=\log x$ for $d=2$. Thus

$$P[E_{j+1} \mid E_j] \leqslant \exp[-ck_R^{-1} h_d(j)], \qquad j<n \tag{45}$$

By induction,

$$(k_R+1)^{-n} c_n(\mathscr{G}_R)=P[E_n] \leqslant \exp\left[-ck_R^{-1} \sum_{j=1}^{n-1} g_d(j)\right] \tag{46}$$

Therefore by (1), for $R=R_4(n, d)$,

$$\begin{aligned}\mu(\mathscr{G}_R) &\leqslant c_n(\mathscr{G}_R)^{1/n}\\ &\leqslant (k_R+1)\exp\left[-ck_R^{-1}(1/n)\sum_{j=1}^{n-1} g_d(j)\right]\\ &\leqslant k_R \exp(k_R^{-1})\exp[-\text{const}\times k_R^{-1} g_d(n)]\\ &\leqslant k_R[1-\text{const}\times k_R^{-1} g_d(n)] \qquad [\text{using } (42)]\end{aligned} \tag{47}$$

Since we set $R=R_4$, by (42) we have $n=\text{const}\times R^{2d/(4+d)}$. Therefore (47) implies the bounds (10) and (11). ∎

3. OTHER MODELS

3.1. Lattice Animals

A lattice animal in $\mathscr{G}$ is a finite connected subgraph of $\mathscr{G}$. Let $a_n(\mathscr{G})$ denote the number of lattice animals in $\mathscr{G}$ with n edges and with the origin as a vertex. Again by using supermultiplicativity,[13] one can show that the limit $\lambda_a(\mathscr{G}) := \lim_{n\to\infty}(a_n(\mathscr{G}))^{1/n}$ exists. Clearly, $a_n(\mathscr{G}) \geqslant t_n(\mathscr{G})$. Also, by an argument in Klarner,[13] $\lambda_a(\mathscr{G}) \leqslant \tau(k(\mathscr{G}))$. Therefore, the statement of Theorem 2 still holds with $\lambda(\mathscr{G}_R)$ replaced by $\lambda_a(\mathscr{G}_R)$.

3.2. Models with Excluded Volume Along Bonds

When the typical step size of a self-avoiding walk or a tree on the sites of Z^d is large (as in the spread-out limit), it might be considered more realistic to have an excluded volume effect not only at the sites visited by the walk or tree, but also near the line segments joining successive sites. This can be done via the following models.

Given a sequence $S_0,\ldots, S_n$ in $\mathscr{R}^d$, let $\mathscr{E}_i$ denote the line segment from S_{i-1} to S_i, i.e., let $\mathscr{E}_i := \{\alpha S_{i-1}+(1-\alpha)S_i\colon 0\leqslant\alpha\leqslant 1\}$. For $r\geqslant 0$ let $c_n^r(\mathscr{G})$ denote the number of $\mathscr{G}$-random walk paths $0=S_0, S_1, S_2,\ldots, S_n$ such that for $1\leqslant i<j\leqslant n$, $\text{dist}(S_j, \mathscr{E}_i)>r$ and $\text{dist}(S_{i-1}, \mathscr{E}_j)>r$.

This counts the number of random walk paths (chains) where each vertex avoids the r-neighborhood of edges other than its immediate neighbors in the chain. A stronger condition is to make each *edge* avoid the r-neighborhood of edges other than its immediate neighbors in the chain. A count of such paths is given by $\hat{c}_n^r(\mathscr{G})$, defined to be the number of $\mathscr{G}$-random walk paths $0 = S_0, S_1, S_2, \ldots, S_n$ such that for $|i-j| > 1$, $\text{dist}(\mathscr{E}_j, \mathscr{E}_i) > r$.

The usual submultiplicative inequalities still hold for these quantities, implying existence of the limits $\mu_r(\mathscr{G}) := \lim(c_n^r(\mathscr{G}))^{1/n}$ and $\hat{\mu}_r(\mathscr{G}) := \lim(\hat{c}_n^r(\mathscr{G}))^{1/n}$. The proof of Theorem 1 can be adapted to give us

$$\mu_r(\mathscr{G}_R) \sim k_R \qquad \text{as} \quad R \to \infty, \qquad r \geqslant 0, \quad d \geqslant 2 \tag{48}$$

It is clear that (48) will not hold for $d = 1$. On the other hand, for $d \geqslant 2$, in the rescaled model on $\mathscr{G}'_R$ the region to be avoided at the $(n+1)$th step is the union of tubes of width r/R around the (rescaled) edges $\mathscr{E}'_i$, $i \leqslant n$, which is small. In addition, the edge $\mathscr{E}'_{n+1}$ must avoid the r/k_R-neighborhoods of the sites visited earlier, but the angle subtended by these regions is also small.

For the case of paths counted by $\hat{c}_n^r(\mathscr{G})$, in $d = 2$ an edge can be hemmed in by a fixed, finite number of earlier edges, no matter how thin the exclusion region around them is. But for $d \geqslant 3$ this cannot happen, and one can show that for any fixed $r \geqslant 0$,

$$\hat{\mu}_r(\mathscr{G}_R) \sim k_R \qquad \text{as} \quad R \to \infty, \qquad d \geqslant 3 \tag{49}$$

Analogous results to (48) and (49) can be obtained for trees. Each tree embedded in $\mathscr{G}$ may be viewed as a collection of sites $z \in Z^d$ and edges $\mathscr{E}$ of $\mathscr{G}$. In an obvious way, one can identify each site with a point in $\mathscr{R}^d$ and each edge with a line segment in $\mathscr{R}^d$. Let $t_n^r(\mathscr{G})$ denote the number of trees θ embedded in $\mathscr{G}$, with n edges and with $0 \in \theta$, such that $\text{dist}(z, \mathscr{E}) > r$ for every site $z \in \theta$ and edge $\mathscr{E} \in \theta$ with z not an endpoint of $\mathscr{E}$. Let $\hat{t}_n^r(\mathscr{G})$ denote the number of trees θ embedded in $\mathscr{G}$, with n edges and with $0 \in \theta$, such that $\text{dist}(\mathscr{E}, \mathscr{E}') > r$ for every pair of edges $\mathscr{E}$ and $\mathscr{E}'$ of θ having no common endpoint.

The limits $\lambda_r(\mathscr{G}) := \lim(t_n^r(\mathscr{G}))^{1/n}$ and $\hat{\lambda}_r(\mathscr{G}) := \lim(\hat{t}_n^r(\mathscr{G}))^{1/n}$ exist, at least when $\mathscr{G} = \mathscr{G}_R$ with $R > r + 1$. The analogous result to (48) is

$$\lambda_r(\mathscr{G}_R) \sim \tau(k_R) \qquad \text{as} \quad R \to \infty, \qquad r \geqslant 0, \quad d \geqslant 2 \tag{50}$$

while the analogous result to (49) is

$$\hat{\lambda}_r(\mathscr{G}_R) \sim \tau(k_R) \qquad \text{as} \quad R \to \infty, \qquad r \geqslant 0, \quad d \geqslant 3 \tag{51}$$

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